Optimal Multi-Period Investment Strategies

Richard Bruynel

National Business Development Manager – Wholesale

BNZ Investment Management

Telephone: 04 382 2697 or 029 402 7490

“Take more investment risk when you are young”

Standard reasons

- Because people near retirement have less time to make up for investment losses.
- Older people value certainty in outcome more than the chance of higher returns.
- People become more risk averse as they get older.

But is this rational?

Example

Consider a saver putting in $1,000 p.a. for 2 years

- There is a range of funds to choose between.

Fund	F3	F4	F5	F6	F7	F8
Expected Return	3%	4%	5%	6%	7%	8%
Standard Deviation of Return	3%	4%	6%	9%	13%	18%

- At the start the saver nominates which fund to invest in each year.

Investment Strategy A

The saver chooses F8 for the first year and F3 for the second year

Expected return after 1 year is 1000 x 1.08 = $1,080

Expected return after 2 years is (1,080 + 1000) x 1.03 = $2,142.40

Investment Strategy B

The saver chooses F6 for the first year and F4 for the second year

Expected return after 1 year is 1000 x 1.06 = $1,060

Expected return after 2 years is (1,060 + 1000) x 1.04 = $2,142.40

Are the Strategies Equivalent?

- Strategies A and B produce the same expected return at the end of 2 years.

But will they have the same standard deviation of return at the end of two years?

Strategy A (F8, F3): Standard Deviation = 195.7

Strategy B (F6, F4): Standard Deviation = 124.8

Assumption: returns in the two years are independent.

What is the Best Strategy

Of all the strategies producing expected return \$2,142.40, which one produces the lowest standard deviation of return after two years?

Analysis of Strategies

	Expected return rate year 1	Expected return rate year 2	Expected fund value	Standard Deviation of fund value
Strategy A	8%	3%	$2,142.40	195.69
Strategy B	6%	4%	$2,142.40	124.74
Constant Fund	4.67%	4.67%	$2,142.40	120.44
Optimal Strategy	5.19%	4.41%	$2,142.40	117.92

The Model Framework - Cashflows

Consider an investment project as a set of planned cashflows over time

Let C_t be the cash flow planned at time t .

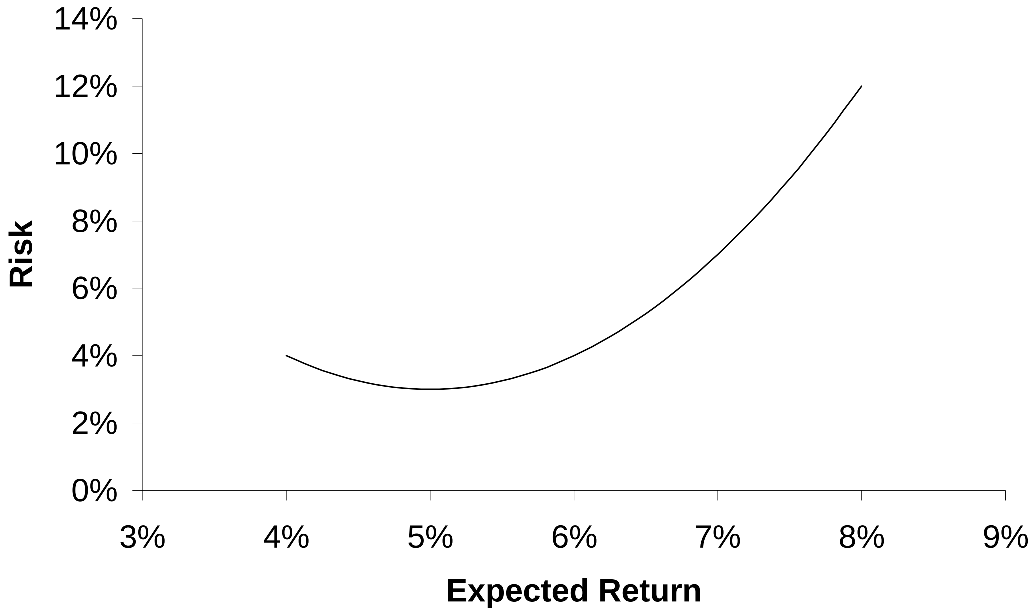
So a 10-year investment project might be:

{1000, 1500, 2000, 2000, 0, 0, -1000, -1000, -1500, -1500}

The Model Framework – Portfolio choice

The available portfolios are all the points on an efficient frontier, given by:

Risk = $\sigma(r)$



The Model Framework – Investment Strategy

At the start of the project the investor can choose what portfolio the accumulated fund should be invested in, for each year of the project.

A 10-year investment strategy might be:

{8%, 6%, 7%, 4%, 5%, 8%, 8%, 7%, 6%, 3%}

Defining the Optimal Investment Strategy

Let the desired expected return be R.

The Optimal Investment Strategy is the minimum variance strategy among all strategies producing expected return R.

Deriving the Optimal Investment Strategy

Main steps

- Find an expression for the expected fund size after *n* periods;

- Find an expression for the variance of the fund size after *n* periods;

- Assume **S** is the optimal investment strategy;

- Vary **S** slightly. As this new strategy is not better than **S**, derive a condition that **S** must satisfy.

The Optimal Investment Strategy

The condition derived is that:

$$\frac{\prod_{k=0}^{t}(1+r_k)}{E_t + C_t}\left(\frac{\partial S_n}{\partial r_t} + \sigma'(r_t)\frac{\partial S_n}{\partial \sigma_t}\right)$$ is constant at all times *t* = 0, 1, 2, ...

E_t is the expected fund size at time *t*

S_n is the second moment of the fund at time *n*

The Optimal Investment Strategy

The recursive expression is:

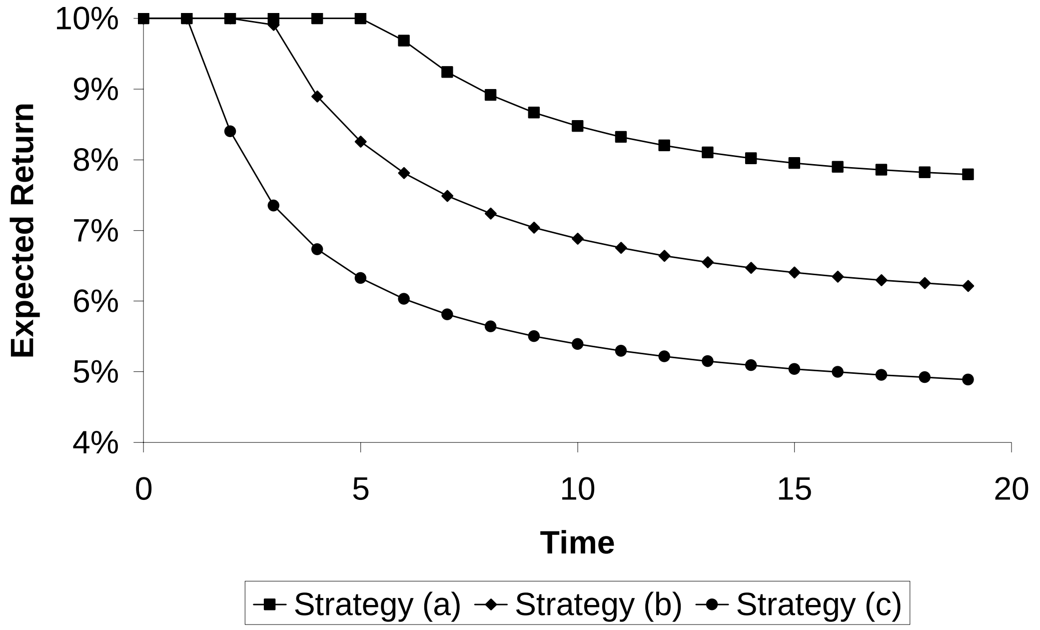
$$g(r_{t+1}) = \frac{\left[\left(1 + r_t + \sigma_t \sigma'(r_t)\right)\left(S_t + 2C_t E_t + C_t^2\right) + C_{t+1}\left(E_t + C_t\right)\right]\left(E_{t+1} + C_{t+1}\right)}{\left(S_{t+1} + 2C_{t+1}E_{t+1} + C_{t+1}^2\right)\left(E_t + C_t\right)}$$

where

$$g(r) = \frac{(1+r)\left(1 + r + \sigma(r)\sigma'(r)\right)}{(1+r)^2 + \sigma^2(r)}$$

Optimal Investment Strategy for a Regular Saver

Investment project is {1, 1, 1, …, 1}



Performance under other criteria

How does the mean/variance optimal investment strategy perform under:

- Different risk measures, e.g. percentiles of the distribution of end values.
- Differing investment assumptions, e.g. correlated returns.

Simulations for 20-year Regular Saver

	Uncorrelated returns		**Annually Correlated returns Correlation = 0.3**	
	Balanced Fund 7% expected return	Mean/Variance Optimal Strategy	Balanced Fund 7% expected return	Mean/Variance Optimal Strategy
Mean	43.87	43.89	43.64	43.71
Standard Deviation	9.34	8.98	12.21	11.80
Median	42.85	42.86	42.99	43.15
20%-ile	35.92	36.29	34.46	34.79
5%-ile	30.34	31.06	27.88	28.61
1%-ile	26.43	27.39	23.09	24.03

Optimal Investment Strategy for a Lump Sum Investor

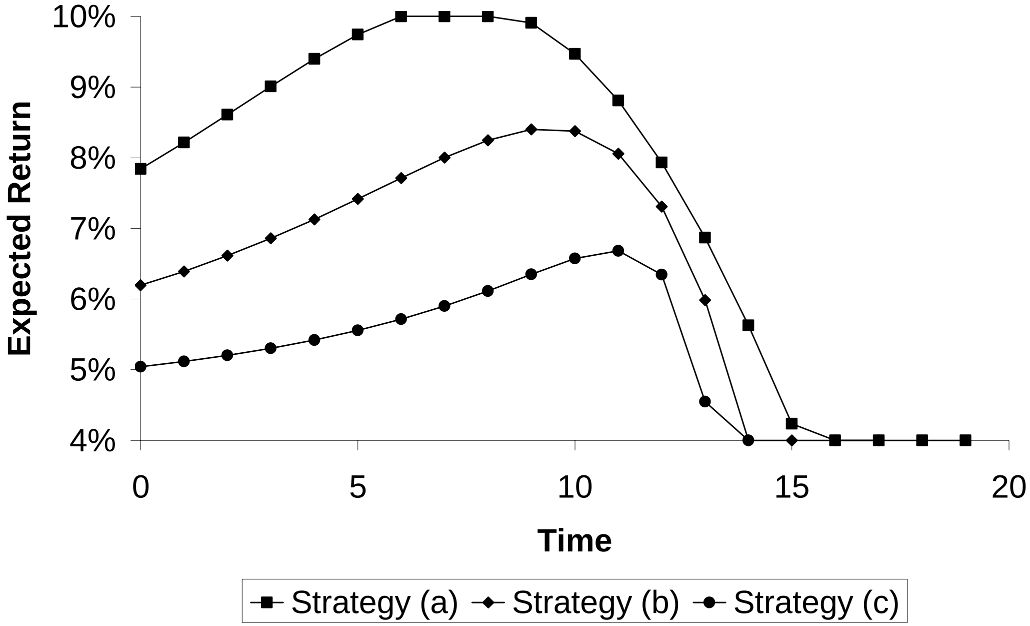
Investment project is {1, 0, 0, …, 0}

The equations all simplify down to give $P_t = P_{t+1}$

The optimal investment strategy is to invest in the same fund the whole time – the Balanced Fund strategy.

Optimal Investment Strategy for a Retiree

Investment project is {10, -1, -1, … -1}



What is the general principle?

Take the greatest investment risk when the fund size is small, invest more conservatively when the fund is larger.

How investors can apply this model

1. Determine the cash flows making up a project.
2. Examine the investment opportunities available.
3. Select the efficient frontier.
4. Identify optimal investment strategies.
5. Choose the strategy having the most acceptable trade-off between risk and return.
6. Each year, invest in a portfolio on the efficient frontier having risk/return levels in line with that suggested by step 5.

Observations

- Every different set of cashflows has a different optimal investment strategy.

- Any non-optimal investment strategy introduces some level of avoidable investment risk, for which there should be no economic return.